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DILATATIONALLY NONLINEAR ELASTIC MATERIALS: (I) SOME THEORY

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by

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ABSTRACT

This paper, which is the first in a two-part study, addresses certain issues concerning the small-strain theory of nonlinear elasticity. It considers isotropic materials which possess a linear response in shear and a nonlinear response in dilatation, and (i) establishes an explicit necessary and sufficient condition for the existence of piecewise homogeneous deformations, (ii) obtains a characterization of the set of all such deformations, (iii) derives an expression for the "driving traction" on a surface of discontinuity in the strain, and finally (iv) discusses the notion of a kinetic law. While the analysis is carried out within a three-dimensional setting, the results are shown to have a particularly simple form when expressed in terms of a certain constitutive function $\Sigma(\epsilon)$. In Part II of this study we examine a specific boundary-value problem.

Sigma

1. Introduction

In this paper, which is the first in a two-part study, we show that certain features of the finite theory of elasticity are also present in the small-strain nonlinear theory; the particular class of constitutive laws that we consider here is one that has been used to model the mechanical response of ceramic composites undergoing supercritical phase transformations. In Part II we will examine a specific boundary-value problem.

A number of recent studies in finite deformation elasticity theory have been concerned with "nonelliptic materials", see for example [1-10]. Such materials are capable of sustaining deformations whose gradient is discontinuous across certain surfaces in the body; this leads to a tremendous lack of uniqueness of solution to boundary-value problems, since the class of functions from among which a solution is sought has to be greatly enlarged to allow for such deformations. Moreover, quasi-static motions of a body composed of such a nominally elastic material can involve a dissipation of mechanical energy at particles located on a moving surface of discontinuity, [9].

Continuum mechanical treatments of stress-induced phase transformations in solids involve such deformations, e.g. [6,7]. In the context of phase transformations, a surface of displacement gradient discontinuity corresponds to a phase boundary separating two different phases of the material, and the aforementioned non-uniqueness might be thought of as arising due to the fact that the classical equations of the continuum theory do not account for the kinetics of the transformation.

In the present study we examine the corresponding issues within the infinitesimal strain theory of nonlinear elasticity. We show that the aforementioned phenomena (of discontinuous, dissipative, non-unique deformations) persist in the infinitesimal strain theory too, suggesting that (in some sense) it is the constitutive nonlinearity rather than the kinematical one that is the principal source of these features.

In this study we restrict attention to the particular class of constitutive laws that were proposed by Budiansky, Hutchinson and Lambropoulos[14] for modeling the mechanical response of certain transforming ceramics. The fracture toughness of these ceramic composites (which contain second phase particles that undergo a phase transformation) was known to be higher than that of the brittle ceramic matrix [11,12,13]. In order to model this phenomenon at the continuum level, Budiansky, Hutchinson and Lambropoulos [14] derived a homogenized constitutive law for such composites using arguments based on the self-consistent method. They argued that since the transformation leads to particles twinned into layers of alternating shear, the average shear associated with the transformation, from a continuum point of view, is essentially zero. Accordingly, they proposed (and studied) a constitutive law with a linear response in shear and a trilinear response in dilatation; see also Silling [15]. It is this class of materials that we will study here (modified to allow the dilatational response to be arbitrary).

Chen and Reyes Moral [16] have experimentally examined the relative importance of shear and dilatation in transforming ceramics, and Lambroupolos [17] has proposed a more general constitutive law that accounts for both of these effects. We do not consider such generalizations here.

In this paper, we first recall the ellipticity conditions for the three-dimensional displacement equations of equilibrium; they are shown to have a particularly simple interpretation in terms of the stress response function of the material in uni-axial deformation, $\Sigma(\epsilon)$. Next, in Section 3, we examine conditions under which a three-dimensional piecewise homogeneous deformation can be sustained by the material, and derive a single necessary and sufficient condition for the existence of piecewise homogeneous deformations. This condition too is expressed in a particularly simple form in terms of $\Sigma(\epsilon)$; in addition to providing information on existence, it also allows us to characterize the set of all possible piecewise homogeneous equilibrium states.

As shown by Knowles[9], when the theory of finite elasticity is broadened to allow for equilibrium fields with discontinuous displacement gradients, the usual balance between the rate of external work and the rate of storage of elastic energy during a quasi-static motion no longer holds. Instead, one finds that mechanical energy may be dissipated at points on the surfaces of discontinuity. This in turn permits one to introduce the notion of a "driving traction" which may be viewed as a normal traction that the body applies to the surface of discontinuity at each of its points. In Section 4 we observe that a dissipation of mechanical energy can also occur in the small-strain theory of elasticity, and we derive an explicit expression for the driving traction in the case of the aforementioned materials; this too may be simply expressed in terms of $\Sigma(\epsilon)$.

The stress response function in uni-axial deformation $\Sigma(\epsilon)$ plays such a

visible role in all of these results because (as shown in Section 3) the local deformations on the two sides of a surface of discontinuity differ from each other by precisely a uni-axial stretch in the direction normal to that surface.

In Section 5 we briefly discuss the need for additional constitutive information in order to complete the theory. As discussed there, this might, for example, take the form of a "kinetic law" which relates the driving traction on the surface of discontinuity to its velocity of propagation. The "flow rule" utilized by Budiansky et al[14] is equivalent to a particular kinetic law as will be discussed more fully in Part II.

The results in this paper pertaining to the existence of piecewise homogeneous deformations (in three-dimensions) have a similar form to analogous results for isotropic, incompressible elastic materials undergoing finite <u>plane</u> deformations, [20]. Likewise, the (three-dimensional) driving traction formula here is similar to the corresponding formulae for finite <u>plane</u> and <u>anti-plane</u> deformations [20,21]. A discussion of kinetic relations in the particular setting of the one-dimensional theory of bars was given in [22].

Finally, in Section 6 we observe that the driving force on the tip of a crack is generally affected by the presence of a surface of strain discontinuity, <u>even if the crack is stationary</u>; see [14]. A relationship between the far field value of the J-integral, the near-tip value of J and the resultant driving force on the surface of discontinuity is derived.

2. Preliminaries.

Consider an elastic body occupying a region R of three-dimensional space. Let $\underset{\sim}{x}$ be the position vector of a particle in R and let $\underset{\sim}{u}(\underset{\sim}{x})$ be its displacement. Suppose that there is a smooth surface S which lies in R, such that the displacement field is continuous on R and twice continuously differentiable on R-S; $\nabla \underset{\sim}{u}$ may suffer a finite jump discontinuity across S. Let $\underset{\sim}{H}$, $\underset{\sim}{\epsilon}$ and $\underset{\sim}{e}$ denote the displacement gradient tensor, the infinitesimal strain tensor and the strain deviator respectively:

$$\left.\begin{aligned} H_{ij} &= u_{i,j}, \\ \epsilon_{ij} &= 1/2(\, u_{i,j} + u_{j,i}), \\ e_{ij} &= \epsilon_{ij} - 1/3\ \epsilon_{kk}\, \delta_{ij}, \end{aligned}\right\} \quad \text{for } \underset{\sim}{x} \in \text{R-S}. \tag{2.1}$$

Displacement continuity across S requires

$$[[u_{i,j}]]\ell_j = 0 \text{ for } \underset{\sim}{x} \in S \tag{2.2}$$

for all vectors $\underset{\sim}{\ell}$ that are tangential to S at $\underset{\sim}{x}$; $[[\cdot]]$ indicates the jump across the surface S. Finally, let $\Delta(\underset{\sim}{x})$ and $k(\underset{\sim}{x})$ denote the respective strain invariants which represent the dilatation and shear at a particle $\underset{\sim}{x}$:

$$\left.\begin{aligned} \Delta &= \operatorname{tr} \underset{\sim}{\epsilon}, \\ k &= [2\operatorname{tr}(\underset{\sim}{e}^2)]^{1/2}, \end{aligned}\right\} \quad \text{for } \underset{\sim}{x} \in \text{R-S}. \tag{2.3}$$

Next, let $\underset{\sim}{\sigma}(\underset{\sim}{x})$ be the stress tensor field on R and suppose that $\underset{\sim}{\sigma}(\underset{\sim}{x})$ is continuously differentiable on R-S; $\underset{\sim}{\sigma}$ may suffer a finite jump discontinuity across S. Equilibrium in the absence of body forces requires

$$\sigma_{ij,j} = 0, \quad \sigma_{ij} = \sigma_{ji} \qquad \text{for } \underset{\sim}{x} \in \text{R-S}, \tag{2.4}$$

$$[[\sigma_{ij}]]n_j = 0 \qquad \text{for } \underset{\sim}{x} \in S, \tag{2.5}$$

where $\underset{\sim}{n}$ is a unit normal vector on S. A surface S which carries jump discontinuities in $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\sigma}$ while maintaining displacement and traction continuity is called an <u>equilibrium shock</u> or <u>phase boundary</u>.

Turning to the constitutive law of the material at hand, suppose that it is homogeneous, isotropic and hyperelastic. The elastic potential W then depends on the deformation only through the three principal invariants of strain. A particular case of special interest is that in which W depends only on the shear and dilatational invariants k and Δ:

$$W(\underset{\sim}{\epsilon}) = W(k,\Delta). \tag{2.6}$$

The stress-strain relation $\underset{\sim}{\sigma} = \partial W/\partial\underset{\sim}{\epsilon}$ at a particle $\underset{\sim}{x} \in R\text{-}S$ then specializes to

$$\sigma_{ij} = (2/k)\partial W/\partial k \; \epsilon_{ij} \; + \; (\partial W/\partial\Delta - (2\Delta/3k)\partial W/\partial k)\, \delta_{ij}. \tag{2.7}$$

If the material is such that the mean stress $\sigma_{ii}/3$ depends on the deformation only through the dilatation ϵ_{ii}, one can show using (2.7) that it is necessary and sufficient that (2.6) have the separable form $W(k,\Delta) = f(k) + g(\Delta)$ which can be more conveniently written as

$$W(k,\Delta) = \int_0^k \hat{\tau}(\kappa)d\kappa \; + \int_0^\Delta \hat{\sigma}(\xi)d\xi \quad \text{for } k\geq 0,\; -\infty<\Delta<\infty; \quad \hat{\tau}(0)=\hat{\sigma}(0)=0. \tag{2.8}$$

(Alternatively, one can show that the components of deviatoric stress depend on the deformation solely through the components of deviatoric strain if and only if W has the form (2.8).) The constitutive functions $\hat{\tau}(k)$ and $\hat{\sigma}(\Delta)$ may be readily interpreted as follows: in a simple shear deformation $u_1 = kx_2$, $u_2 = 0$, $u_3 = 0$, the shear stress component σ_{12} is

found from (2.8), (2.7) to be $\sigma_{12} = \hat{\tau}(k)$; in a pure dilatational deformation $u_i = (\Delta/3)x_i$, one finds that $\sigma_{ii}/3 = \hat{\sigma}(\Delta)$. Thus the function $\hat{\tau}(k)$ is the <u>shear stress response function of the material in simple shear</u>, while the function $\hat{\sigma}(\Delta)$ is the <u>mean stress response function of the material in pure dilatation</u>.

Finally, we further specialize (2.8) to the case in which the shear stress response in simple shear is linear: $\hat{\tau}(k) = \mu k$. This is motivated by the fact that such constitutive relations appear to be of interest in the continuum mechanical modeling of certain ceramic composites containing particles which undergo stress induced phase transformations, (see Budiansky et al[14]). Thus, in this study we consider materials characterized by an elastic potential

$$W(k,\Delta) = (\mu/2)k^2 + \int_0^{\Delta} \hat{\sigma}(\xi)d\xi \quad \text{for } k\geq 0,\ -\infty<\Delta<\infty, \tag{2.9}$$

where μ (>0) is the infinitesimal shear modulus of the material. The stress-strain relation (2.7) now specializes to

$$\sigma_{ij} = 2\mu\, \epsilon_{ij} + (\hat{\sigma}(\Delta) - 2\mu\Delta/3)\delta_{ij}. \tag{2.10}$$

The bulk modulus of the material (2.9) is

$$B(\Delta) = \hat{\sigma}(\Delta)/\Delta \quad \text{for } -\infty<\Delta<\infty. \tag{2.11}$$

It is useful for later purposes to consider the response of this body in a uni-axial deformation $u_1 = \epsilon x_1$, $u_2 = u_3 = 0$. From (2.10) one gets $\sigma_{11} = \Sigma(\epsilon)$ where

$$\Sigma(\epsilon) = \hat{\sigma}(\epsilon) + 4\mu\epsilon/3 \quad \text{for } -\infty<\epsilon<\infty; \tag{2.12}$$

$\Sigma(\epsilon)$ is the <u>stress response function of the material in uni-axial deformation</u>.

The displacement equations of equilibrium for the class of materials under discussion here are, by (2.10), (2.4), (2.1), (2.3)

$$c_{ijkl}(\underset{\sim}{\epsilon})u_{k,jl} = 0 \quad \text{for } \underset{\sim}{x} \in R\text{-}S, \tag{2.13}$$

where

$$c_{ijkl}(\underset{\sim}{\epsilon}) = \mu(\delta_{ik}\delta_{lj} + \delta_{kj}\delta_{li}) + (\hat{\sigma}'(\Delta) - 2\mu/3)\delta_{ij}\delta_{kl}. \tag{2.14}$$

The system of partial differential equations (2.13) is said to be (strongly) <u>elliptic</u> at a solution $\underset{\sim}{u}$ and at a point $\underset{\sim}{x} \in R\text{-}S$ if

$$c_{ijkl}(\underset{\sim}{\epsilon}(\underset{\sim}{x}))\, m_i n_j m_k n_l > 0 \tag{2.15}$$

for all unit vectors $\underset{\sim}{m}$ and $\underset{\sim}{n}$. It is not difficult to show from (2.14), (2.15) and $\mu > 0$ that (strong) ellipticity prevails if and only if

$$\hat{\sigma}'(\Delta(\underset{\sim}{x})) > -4\mu/3 \tag{2.16}$$

where $\Delta(\underset{\sim}{x}) = \epsilon_{kk}(\underset{\sim}{x})$ is the dilatation associated with the given deformation at the point under consideration. Observe from (2.12) that this ellipticity condition can be expressed simply in terms of the stress response function in uni-axial deformations as

$$\Sigma'(\Delta(\underset{\sim}{x})) > 0. \tag{2.17}$$

Thus the ellipticity of the governing equations is directly related to the invertibility of the stress response function in uni-axial deformations. If Σ fails to be monotonically increasing on $-\infty < \epsilon < \infty$, ellipticity will be lost at some deformation. If $\Sigma'(\epsilon) > 0$ for all ϵ, we say that the material is elliptic. We assume throughout that $\Sigma'(0) > 0$ so that ellipticity prevails at the undeformed state; since $\Sigma'(0) = \kappa+4\mu/3$ where κ is the infinitesimal bulk modulus, this, together with $\mu > 0$, are the usual ellipticity conditions of linear elasticity.

3. Piecewise homogeneous displacement fields.

Not all homogenous, isotropic elastic materials characterized by the constitutive relation (2.10) can sustain deformations with discontinuous strains. In this section, we determine a simple necessary and sufficient condition on the material which determines whether or not it can sustain piecewise homogeneous deformations of this type. In addition, for materials that can sustain such deformations, we obtain a characterization (in a certain sense) of the entire collection of possible piecewise homogeneous deformations.

We now consider the special case in which R coincides with all of (x_1,x_2,x_3)-space, S is a plane through the origin, and the displacement gradient is constant on each side of S. Let $\underset{\sim}{n}$ be a unit vector normal to the plane S, and let $\overset{+}{R}$, $\overset{-}{R}$ be the two open half-spaces into which S divides R with the normal $\underset{\sim}{n}$ pointing into $\overset{+}{R}$. The field equations (2.13) will then be trivially satisfied in R -S, and all that remains to be fulfilled are

the jump conditions (2.2) and (2.5).

Consider the piecewise homogeneous displacement field

$$\mathbf{u} = \begin{cases} \overset{+}{\mathbf{H}}\,\mathbf{x} & \text{for } \mathbf{x}\in \overset{+}{R}\,, \\ \overset{-}{\mathbf{H}}\,\mathbf{x} & \text{for } \mathbf{x}\in \overset{-}{R}\,, \end{cases} \tag{3.1}$$

where the displacement gradient tensors $\overset{+}{\mathbf{H}}$ and $\overset{-}{\mathbf{H}}$ are constant and distinct:

$$\overset{+}{\mathbf{H}} \neq \overset{-}{\mathbf{H}}. \tag{3.2}$$

Define $\overset{+}{\boldsymbol{\epsilon}}$, $\overset{-}{\boldsymbol{\epsilon}}$, $\overset{+}{\Delta}$ and $\overset{-}{\Delta}$ by

$$\overset{+}{\epsilon}_{ij} = 1/2\ (\overset{+}{H}_{ij} + \overset{+}{H}_{ji}), \qquad \overset{-}{\epsilon}_{ij} = 1/2\ (\overset{-}{H}_{ij} + \overset{-}{H}_{ji}), \tag{3.3}$$

$$\overset{+}{\Delta} = \overset{+}{\epsilon}_{kk}, \qquad \overset{-}{\Delta} = \overset{-}{\epsilon}_{kk}. \tag{3.4}$$

The displacement field (3.1) will be continuous across S if and only if

$$\overset{+}{H}_{ij}\,\ell_j = \overset{-}{H}_{ij}\,\ell_j \quad \text{for all unit vectors } \boldsymbol{\ell} \text{ normal to } \mathbf{n}, \tag{3.5}$$

while by (2.5), (2.10) the tractions will be continuous across S if and only if

$$2\mu\,\overset{+}{\epsilon}_{ij}n_j + (\hat{\sigma}(\overset{+}{\Delta}) - 2\mu\overset{+}{\Delta}/3)n_i = 2\mu\,\overset{-}{\epsilon}_{ij}n_j + (\hat{\sigma}(\overset{-}{\Delta}) - 2\mu\overset{-}{\Delta}/3)n_i. \tag{3.6}$$

Given a tensor $\overset{+}{\mathbf{H}}$, the <u>shock problem</u> consists of finding a tensor $\overset{-}{\mathbf{H}}$ and a unit vector $\mathbf{n}$ such that (3.5) and (3.6) (with (3.3),(3.4)) hold.

We first establish a necessary condition which must hold if the shock problem is to have a solution. To this end, suppose that given $\overset{+}{\mathbf{H}}$, there is

a tensor $\overset{-}{\mathbf{H}}$ and a unit vector $\mathbf{n}$ such that (3.3)-(3.6) hold. It can be readily shown that (3.5) holds if and only if there exists a vector $\mathbf{a}$ such that

$$\overset{-}{H}_{ij} = \overset{+}{H}_{ij} + a_i n_j. \tag{3.7}$$

Thus (3.3), (3.4), (3.7) yield

$$\overset{-}{\epsilon}_{ij} = \overset{+}{\epsilon}_{ij} + 1/2(a_i n_j + a_j n_i), \tag{3.8}$$

$$\overset{-}{\Delta} = \overset{+}{\Delta} + a_i n_i. \tag{3.9}$$

Turning next to the requirement (3.6) and multiplying it by the components ℓ_i of any unit vector normal to $\mathbf{n}$ gives

$$\overset{+}{\epsilon}_{ij} \ell_i n_j = \overset{-}{\epsilon}_{ij} \ell_i n_j, \tag{3.10}$$

which in view of (3.8) simplifies to

$$a_i \ell_i = 0. \tag{3.11}$$

Since this must hold for all unit vectors $\boldsymbol{\ell}$ in the plane S, it follows that $\mathbf{a}$ is parallel to $\mathbf{n}$:

$$\mathbf{a} = \alpha \mathbf{n}. \tag{3.12}$$

By (3.12), (3.9),

$$\alpha = \overset{-}{\Delta} - \overset{+}{\Delta}. \tag{3.13}$$

Moreover (3.8) can be written, in view of (3.12), as

$$\overset{-}{\epsilon}_{ij} = \overset{+}{\epsilon}_{ij} + \alpha n_i n_j. \tag{3.14}$$

Finally, multiply the traction continuity condition (3.6) by n_i and use (3.14), (3.13) to obtain

$$\hat{\sigma}(\overset{+}{\Delta}) + 4\mu\overset{+}{\Delta}/3 = \hat{\sigma}(\overset{-}{\Delta}) + 4\mu\overset{-}{\Delta}/3, \tag{3.15}$$

which, in terms of the uni-axial deformation response function $\Sigma(\Delta)$, reads

$$\Sigma(\overset{+}{\Delta}) = \Sigma(\overset{-}{\Delta}). \tag{3.16}$$

Next we will show that if the (necessary) condition (3.16) holds, then this in fact guarantees the existence of a solution to the shock problem. In order to show this, suppose that $\overset{+}{\underset{\sim}{H}}$ is a given tensor. Define $\overset{+}{\Delta}$ by $(3.4)_1$, $(3.3)_1$. If there exists a number $\overset{-}{\Delta}$ ($\neq \overset{+}{\Delta}$) such that (3.16) holds, then (for <u>each</u> arbitrary unit vector $\underset{\sim}{n}$) we can define α by (3.13), $\underset{\sim}{a}$ by (3.12) and $\overset{-}{\underset{\sim}{H}}$ by (3.7). It may be readily verified that these tensors $\overset{+}{\underset{\sim}{H}}$ and $\overset{-}{\underset{\sim}{H}}$ automatically satisfy the requirements (3.5), (3.6) of displacement and traction continuity across the plane with unit normal $\underset{\sim}{n}$. Thus we have the following result:

<u>Proposition</u>: Given a tensor $\overset{+}{\underset{\sim}{H}}$, there exists an associated piecewise homogeneous equilibrium shock if and only if there is a number $\overset{-}{\Delta}$ ($\neq \overset{+}{\Delta} = \overset{+}{H}_{kk}$) such that (3.16) holds.

When the constitutive law is such that the stress response function $\Sigma(\epsilon)$ is monotonically increasing (in which case the material is elliptic) we see from the preceding proposition that the material cannot sustain a piecewise homogeneous deformation. On the other hand, if the material is such that $\Sigma'(\Delta) \leq 0$ on some interval, then since $\Sigma'(0) > 0$, it follows that piecewise homogeneous deformations will exist for suitably chosen values of $\overset{+}{\underset{\sim}{H}}$.

In addition to providing information concerning the existence of a piecewise homogeneous deformation associated with the given displacement gradient $\overset{+}{\underset{\sim}{H}}$, the preceding result also permits us to characterize the set of all such deformations which can be associated with that $\overset{+}{\underset{\sim}{H}}$: it states that for every number $\bar{\Delta}$ for which (3.16) holds, and for all choices of the unit normal vector $\underset{\sim}{n}$, one can construct an acceptable $\overset{-}{\underset{\sim}{H}}$. Let Ξ denote the following set in the $(\overset{+}{\Delta}, \overset{-}{\Delta})$-plane:

$$\Xi = \{ (\overset{+}{\Delta}, \overset{-}{\Delta}) \mid \Sigma(\overset{+}{\Delta}) = \Sigma(\overset{-}{\Delta}),\ \overset{+}{\Delta} \neq \overset{-}{\Delta} \}. \tag{3.17}$$

According to the preceding proposition, given a displacement gradient tensor $\overset{+}{\underset{\sim}{H}}$, the associated shock problem has a solution if and only if there is a number $\bar{\Delta}$ such that $(\overset{+}{\Delta}, \overset{-}{\Delta}) \in \Xi$ where $\overset{+}{\Delta} = \overset{+}{H}_{kk}$; moreover, all tensors $\overset{-}{\underset{\sim}{H}}$ that can be connected to $\overset{+}{\underset{\sim}{H}}$ by a shock are generated by all numbers $\bar{\Delta}$ for which $(\overset{+}{\Delta}, \overset{-}{\Delta}) \in \Xi$. The set Ξ characterizes the collection of all possible shocks. A sketch of the curve Ξ in the $(\overset{+}{\Delta}, \overset{-}{\Delta})$-plane, corresponding to a particular class of materials, will be given in Section 5.

Finally, we note that according to (3.7), (3.12), (3.13) the displacement gradient tensors $\overset{+}{\underset{\sim}{H}}$ and $\overset{-}{\underset{\sim}{H}}$ are related by

$$\overset{-}{H}_{ij} = \overset{+}{H}_{ij} + (\overset{-}{\Delta} - \overset{+}{\Delta})\, n_i n_j; \tag{3.18}$$

this implies that the deformation on $\overset{-}{R}$ is equivalent to the deformation on $\overset{+}{R}$ together with a uni-axial stretch in the direction normal to the shock surface. This is presumably the reason why the stress response function in uni-axial deformation $\Sigma(\epsilon)$ plays such a central role in the preceding (and subsequent) results.

4. Driving Traction

We now consider a quasi-static motion of the body and let $\underset{\sim}{u}(\cdot,t)$, $t_0 \le t \le t_1$, be a one-parameter family of solutions of the displacement equations of equilibrium (2.13) of the type described in Section 2. Let $S_t \subset R$ be the family of shocks associated with this motion, and assume that the particle velocity $\underset{\sim}{v}(\underset{\sim}{x},t) = \partial\underset{\sim}{u}(\underset{\sim}{x},t)/\partial t$ exists and is continuous in $(\underset{\sim}{x},t)$ for $\underset{\sim}{x} \in R\text{-}S_t$, $t_0 \le t \le t_1$, and that $\underset{\sim}{v}$ is piecewise continuous on $R \times [t_0,t_1]$.

Let $d(t)$ denote the difference between the rate of external work (on any fixed regular region $\Pi \subset R$) and the rate at which elastic energy is being stored (in Π):

$$d(t) = \int_{\partial\Pi} \sigma_{ij} n_j v_i \, dA - \frac{d}{dt}\int_{\Pi} W(\underset{\sim}{\epsilon}) \, dV, \qquad t_0 \le t \le t_1; \tag{4.1}$$

$d(t)$ is the rate of dissipation of mechanical energy in the region Π. By adapting to the present small-strain theory the analysis given by Knowles [9], one can show that $d(t)$ may be written as

$$d(t) = \int_{S_t \cap \Pi} f \, \underset{\sim}{n}\cdot\underset{\sim}{V} \, dA, \tag{4.2}$$

where $f(\underset{\sim}{x},t)$ is defined by

$$f = [[\underset{\sim}{P}]]_{-}^{+}\underset{\sim}{n} \cdot \underset{\sim}{n}, \quad \text{for} \quad \underset{\sim}{x} \in S_t, \quad t_0 \le t \le t_1, \tag{4.3}$$

$\underset{\sim}{P}(\underset{\sim}{x},t)$ is the energy-momentum tensor

$$P_{ij} = W(\underset{\sim}{\epsilon})\delta_{ij} - \sigma_{kj} H_{ki} \quad \text{for } \underset{\sim}{x} \in R\text{-}S_t,\ t_0 \le t \le t_1, \tag{4.4}$$

and $\underset{\sim}{V}(\underset{\sim}{x},t)$ is the velocity of a point on the moving surface S_t. If the motion happens to be smooth, $\underset{\sim}{P}(\cdot,t)$ will be continuous across S_t and so (4.2), (4.3) gives $d(t) = 0$ for $t_0 \le t \le t_1$. . In general however the dissipation rate $d(t) \neq 0$ whenever Π intersects S_t.

Combining (4.1) with (4.2) yields

$$\int_{\partial\Pi} \underset{\sim}{\sigma}\underset{\sim}{n}\cdot\underset{\sim}{v}\ dA + \int_{S_t\cap\Pi} (-f\underset{\sim}{n})\cdot\underset{\sim}{V}\ dA = \frac{d}{dt}\int_{\Pi} W(\underset{\sim}{\epsilon})\ dV, \qquad t_0 \le t \le t_1, \tag{4.5}$$

which may be viewed as a work-energy identity. It states that the sum of the rates at which work is being done on Π by the external forces and the phase boundary S_t balances the rate at which energy is being stored in Π. Accordingly, $-f\underset{\sim}{n}$ may be thought of as the traction applied by the surface S_t on the body, or equivalently, $+f\underset{\sim}{n}$ can be viewed as a "driving traction" exerted on the phase boundary S_t by the surrounding material; the scalar f determines the magnitude of this traction. The expression (4.3) (with (4.4)) is a special case of a formula given by Eshelby[23]; see also Eshelby[24], Rice[25].

If we postulate that at each instant, the rate of storage of energy in Π cannot exceed the rate of external work on Π, then we must require the dissipation rate $d(t)$ to be non-negative for all sub-regions Π and all instants t. Thus, by (4.2),

$$f\ V_n \ge 0 \quad \text{for } \underset{\sim}{x} \in S_t,\ t_0 \le t \le t_1, \tag{4.6}$$

where V_n is the normal velocity of a point on the surface S_t :

$$V_n = \underset{\sim}{V} \cdot \underset{\sim}{n} \quad \text{for } \underset{\sim}{x} \in S_t, \quad t_0 \le t \le t_1. \tag{4.7}$$

Alternatively, the <u>dissipation inequality</u> (4.6) can be shown to be a consequence of the second law of thermodynamics under isothermal conditions; see [9]. In general, given an equilibrium state, the inequality (4.6) restricts the direction in which the surface S_t may move in a quasi-static motion commencing from this state.

A particularly simple expression for the driving traction can be derived in the case of materials characterized by the special elastic potential (2.9). First, from (3.7), (3.12) one has

$$[[H_{ki}]]_{-}^{+} = -\alpha n_i n_k. \tag{4.8}$$

Next, in view of (2.10), (2.12), (4.8) and the continuity of traction,

$$[[\sigma_{kj} H_{ki} n_i n_j]]_{-}^{+} = -\alpha\left(2\mu \overset{+}{\epsilon}_{ij} n_i n_j - 2\mu \overset{+}{\Delta} + \Sigma(\overset{+}{\Delta}) \right). \tag{4.9}$$

However from (2.3), (2.1) and (3.14) one obtains

$$[[k^2]]_{-}^{+} = -4\alpha \overset{+}{\epsilon}_{ij} n_i n_j - 2\alpha^2 + 2\alpha(\overset{+}{\Delta} + \overset{-}{\Delta})/3, \tag{4.10}$$

which can be used to eliminate the term $\overset{+}{\epsilon}_{ij} n_i n_j$ from (4.9) to give

$$[[\sigma_{kj} H_{ki} n_i n_j]]_{-}^{+} = \mu(\overset{+}{k}{}^2 - \overset{-}{k}{}^2)/2 + 2\mu(\overset{-}{\Delta}{}^2 - \overset{+}{\Delta}{}^2)/3 - \alpha\Sigma(\overset{+}{\Delta}). \tag{4.11}$$

Finally, since (2.9) and (2.12) provide

$$W(\underset{\sim}{\epsilon}) = \mu k^2/2 - 2\mu\Delta^2/3 + \int_0^{\Delta} \Sigma(\xi)\, d\xi, \tag{4.12}$$

equations (4.3), (4.4), (4.11), (4.12) and (3.13) yield the desired expression

$$f = \int_{\overset{-}{\Delta}}^{\overset{+}{\Delta}} \Sigma(\Delta)\, d\Delta \;-\; \Sigma(\overset{+}{\Delta})\,(\overset{+}{\Delta} - \overset{-}{\Delta}) \quad \text{for } \underset{\sim}{x}\in S_t, \quad t_0\le t\le t_1, \tag{4.13}$$

for the driving traction. In the finite theory, formulae of this general form have been derived in the special case of "normal shocks" in plane and anti-plane finite deformations of isotropic, incompressible elastic solids, [19,20].

It is useful to write (4.13) as

$$f = F(\overset{+}{\Delta}, \overset{-}{\Delta}) \quad \text{for } \underset{\sim}{x}\in S_t, \quad t_0\le t\le t_1 , \tag{4.14}$$

where F is the function defined on the set Ξ by

$$F(\overset{+}{\Delta}, \overset{-}{\Delta}) = \int_{\overset{-}{\Delta}}^{\overset{+}{\Delta}} \Sigma(\Delta)\, d\Delta - \Sigma(\overset{+}{\Delta})(\overset{+}{\Delta} - \overset{-}{\Delta}) \quad \text{for } (\overset{+}{\Delta}, \overset{-}{\Delta})\in \Xi. \tag{4.15}$$

By (4.14), the driving traction f at a point on the phase boundary S_t depends only on the local dilatations $\overset{+}{\Delta}$, $\overset{-}{\Delta}$ on the two sides of S_t; f does not depend on the amounts of shear $\overset{+}{k}$, $\overset{-}{k}$, nor on the orientation of S_t. Moreover, in view of (4.15) and (3.16), the value of f may be interpreted geometrically as the difference between the area under the uni-axial deformation stress-strain curve between $\overset{-}{\Delta}$ and $\overset{+}{\Delta}$, and the area of the rectangle on the same base with height $\Sigma(\overset{+}{\Delta})$.

5. Kinetic Law. An example.

In Part II, we will present an example which shows that boundary-value problems formulated in the conventional manner, for materials characterized by (2.9), may suffer from a tremendous lack of uniqueness. This is known to be the case in the finite theory as well (e.g. [18,19]). This non-uniqueness suggests that the theory, as formulated, is deficient, and that it ought to be supplemented with additional constitutive information. One way in which to implement this is to postulate a constitutive relation, or "kinetic law", which applies to particles on S_t, and relates the driving traction f to the normal velocity of propagation V_n of the phase boundary.

In order to formalize this, let f_M and f_m be the supremum and infemum of the function $F(\overset{+}{\Delta}, \overset{-}{\Delta})$ on the set Ξ. Then, one might suppose that there is a constitutive function $V(\cdot)$ defined on $[f_m, f_M]$ such that

$$V_n = V(f) \quad \text{on } S_t, \quad t_0 \le t \le t_1. \tag{5.1}$$

In order to conform to the dissipativity inequality (4.7), V must be such that

$$f\,V(f) \ge 0 \quad \text{for } f \in [f_m, f_M]. \tag{5.2}$$

The form (5.1) is, of course, merely an example of a class of kinetic laws that might be imposed; it could be generalized to include dependence on other local variables as well so that, for example, a kinetic law might read $V_n = V(\overset{+}{\Delta}, \overset{-}{\Delta})$, where the constitutive function V is defined on Ξ.

In order to illustrate this (and some of the preceding results) we now choose the dilatational stress response function in the constitutive law

(2.10) to be as follows:

$$\hat{\sigma}(\Delta) = \begin{cases} \beta\Delta & \text{for } 0 \le \Delta \le \Delta_M, \\ \beta\Delta + \sigma_T(\Delta-\Delta_M)/(\Delta_m-\Delta_M) & \text{for } \Delta_M \le \Delta \le \Delta_m, \\ \beta\Delta + \sigma_T & \text{for } \Delta \ge \Delta_m; \end{cases} \tag{5.3}$$

β, Δ_m, Δ_M and σ_T are material constants such that

$$\left.\begin{array}{l} \beta>0, \quad \Delta_m>\Delta_M>0, \quad \sigma_T<0, \\ \beta\Delta_m + \sigma_T > 0, \\ (\Delta_m-\Delta_M)(\beta+4\mu/3) < -\sigma_T. \end{array}\right\} \tag{5.4}$$

The second condition in (5.4) implies that $\hat{\sigma}(\Delta_m) > 0$, while $(5.4)_3$ ensures that the system of equations (2.13) is non-elliptic when $\Delta_m < \Delta(\underset{\sim}{x}) < \Delta_M$. In this example we will confine attention to the range $\Delta > 0$ and consequently we have left $\hat{\sigma}(\Delta)$ undefined for negative values of its argument. The specific constitutive law (5.3), (5.4) is the one considered by Budiansky, Hutchinson and Lambropoulos [14] in the case of super-critical transformations. As shown in Figure 1, as the dilatation increases, the mean stress first rises linearly to a maximum value $\sigma_M = \beta\Delta_M$, it then declines linearly to the value $\sigma_m = \beta\Delta_m + \sigma_T$, and finally rises again with the initial slope β.

The response function of this material in uni-axial deformations is given by (5.3), (2.12) as

$$\Sigma(\epsilon) = \begin{cases} \alpha\epsilon & \text{for } 0 \le \epsilon \le \Delta_M, \\ \alpha\epsilon + \sigma_T(\epsilon-\Delta_M)/(\Delta_m-\Delta_M) & \text{for } \Delta_M \le \epsilon \le \Delta_m, \\ \alpha\epsilon + \sigma_T & \text{for } \epsilon \ge \Delta_m, \end{cases} \tag{5.5}$$

where we have set

$$\alpha = \beta + 4\mu/3. \tag{5.6}$$

Finally, we introduce the following additional notation which pertains to certain special points on the stress-strain curve shown in Figure 1:

$$\left.\begin{array}{ll} \Delta_{m1} = \Delta_m + \sigma_T/\alpha, & \Delta_{M3} = \Delta_M - \sigma_T/\alpha, \\ \Delta_{o1} = (\Delta_m+\Delta_M)/2 + \sigma_T/2\alpha, & \Delta_{o3} = (\Delta_m+\Delta_M)/2 - \sigma_T/2\alpha. \end{array}\right\} \tag{5.7}$$

Note that the straight lines which join (Δ_M, σ_M) to $(\Delta_{M3}, \hat{\sigma}(\Delta_{M3}))$, $(\Delta_{o1}, \hat{\sigma}(\Delta_{o1}))$ to $(\Delta_{o3}, \hat{\sigma}(\Delta_{o3}))$, and $(\Delta_{m1}, \hat{\sigma}(\Delta_{m1}))$ to (Δ_m, σ_m), each have the same slope $-4\mu/3$; see (3.15) for the significance of this. Moreover, Δ_{o1} and Δ_{o3} are seen to obey the conditions

$$\left.\begin{array}{lll} & \Sigma(\Delta_{o1}) = \Sigma(\Delta_{o3}) = (\Sigma_m+\Sigma_M)/2, & \\ \text{where} & \Sigma_m = \Sigma(\Delta_m), & \Sigma_M = \Sigma(\Delta_M). \end{array}\right\} \tag{5.8}$$

The set Ξ for this material, which characterizes the complete set of possible shocks in the $(\overset{+}{\Delta}, \overset{-}{\Delta})$-plane, may be readily found from (3.17), (5.5). It consists of the points <u>on</u> the polygon ABCDEFA shown in Figure 2, except for the vertices A and D which lie on the line $\overset{+}{\Delta} = \overset{-}{\Delta}$.

We turn next to equation (4.15) which defines the driving traction function F on this set Ξ. Explicit formulae for F may be readily derived from (4.15), (5.5). For example, when $(\overset{+}{\Delta}, \overset{-}{\Delta}) \in EF$, one finds

$$F(\overset{+}{\Delta}, \overset{-}{\Delta}) = (-\sigma_T/\alpha)\{ \alpha\overset{+}{\Delta} - (\Sigma_m+\Sigma_M)/2\}. \tag{5.9}$$

We do not display the remaining formulae here. It is particularly useful to know the sign of the driving traction, since then the direction of pro-

pagation of the phase boundary is known through the dissipativity inequality (4.6). The sign of F may be read off from (5.9) (and the analogous formulae appropriate to the other points on Ξ); one finds that

$$F(\overset{+}{\Delta}, \overset{-}{\Delta}) = \begin{cases} > 0 & \text{for } (\overset{+}{\Delta}, \overset{-}{\Delta}) \in (AB]+[BP)+(DE]+[EQ), \\ < 0 & \text{for } (\overset{+}{\Delta}, \overset{-}{\Delta}) \in (PC]+[CD)+(QF]+[FA), \\ = 0 & \text{for } (\overset{+}{\Delta}, \overset{-}{\Delta}) = P \text{ or } Q. \end{cases} \tag{5.10}$$

(The symbol (AB] in (5.10) denotes the set of all points on the line AB excluding the end point A but including the point B.) The points P and Q which are associated with zero driving traction are sometimes refered to as "Maxwell states". They are given by

$$(\overset{+}{\Delta}, \overset{-}{\Delta}) = (\Delta_{o3}, \Delta_{o1}) \text{ and } (\Delta_{o1}, \Delta_{o3}), \tag{5.11}$$

where Δ_{o1} and Δ_{o3} were defined in (5.7); see also (5.8). Also, one finds that the driving traction achieves its largest value f_M at B (and also at E) and its smallest value f_m at C (and also at F). These values are

$$f_M = -\sigma_T(\Sigma_M - \Sigma_m)/2\alpha \quad (>0), \tag{5.12}$$

$$f_m = \sigma_T(\Sigma_M - \Sigma_m)/2\alpha \quad (<0). \tag{5.13}$$

Finally, Figure 3 shows an example of a kinetic function V that might be used in the kinetic law (5.1). It is consistent with the admissibility requirement (5.2). In the example which will be discussed in Part II, we will see how in a specific problem, the kinetic relation, together with an initiation criterion, can be used to resolve the non-uniqueness referred to earlier. In that example we will find that the kinetic relation of Figure 3 generally leads to rate-dependent "viscoplasticity-like" response. Two

special cases which lead to reversible, dissipation-free response and to rate-independent plasticity-like respone will also be discussed there.

6. Concluding remark: driving force on a crack-tip

In this section we briefly comment on the driving force on a crack-tip when the crack is contained in a body composed of the material (2.10). For simplicity, suppose that the body is a slab containing a traction-free through-crack (Figure 4) and that the loading is such that the deformation is planar. Suppose further that the body is composed of the material (2.10) with the constitutive function $\Sigma(\epsilon)$ defined by (2.12) being non-monotone. By the analysis in Section 3, deformations of this body can involve shocks. Suppose for definiteness that there is a single (cylindrical) phase boundary S as shown in Figure 4; C is the curve along which S intersects the (x_1,x_2)-plane. The deformation is smooth at all points of the body inside C (excluding points on the crack itself) as well as at all points outside C.

Let Γ_o and Γ_∞ be two closed curves as shown in Figure 4 with Γ_o being entirely within C and Γ_∞ entirely outside. The values of the J-integral associated with these two curves are respectively,

$$J_{tip} = \oint_{\Gamma_o} P_{1\beta}\, n_\beta\, ds, \qquad J_\infty = \oint_{\Gamma_\infty} P_{1\beta}\, n_\beta\, ds, \tag{6.1}$$

where ds denotes arc length, $\underset{\sim}{n}$ is the unit outward normal vector on the appropriate curve, and $P_{\alpha\beta}$ are the components of the energy-momentum

tensor:

$$P_{\alpha\beta} = W\,\delta_{\alpha\beta} - \sigma_{\gamma\beta}\,H_{\gamma\alpha}. \tag{6.2}$$

(Greek subscripts take the values 1 and 2 only.)

The J-integral is path-independent provided the paths of integration do not insect the shock curve C; this, together with the traction-free nature of the crack surface yields the alternate expression

$$J_{tip} = \oint_C \bar{P}_{1\beta}\, n_\beta\, ds, \tag{6.3}$$

where $\bar{P}_{\alpha\beta}$ are the limiting values of $P_{\alpha\beta}$ as a point on C is approached from within. Similarly

$$J_\infty = \oint_C \overset{+}{P}_{1\beta}\, n_\beta\, ds. \tag{6.4}$$

Combining (6.3) and (6.4) gives

$$J_{tip} = J_\infty - \oint_C [[P_{1\beta}]]_-^+\, n_\beta\, ds. \tag{6.5}$$

Next, in view of (6.2), (2.1), displacement continuity (2.2), and traction continuity (2.5), one sees that

$$[[P_{\alpha\beta}]]_-^+ n_\beta\, \ell_\alpha = 0 \quad \text{on C}, \tag{6.6}$$

where $\underset{\sim}{\ell}$ is a unit tangent vector on C. Thus the vector $[[P_{\alpha\beta}]]_-^+ n_\beta$ is normal

to the curve C, and by (4.3),

$$[[P_{\alpha\beta}]]_{-}^{+} n_\beta = f\, n_\beta \quad \text{on } C; \tag{6.7}$$

f is the driving traction on the shock. Finally, combining (6.5) with (6.7) provides the desired expression

$$J_{tip} = J_\infty - \oint_C f\, n_1\, ds. \tag{6.8}$$

Equation (6.8) states that the driving force J_{tip} on the crack-tip equals the difference between J_∞ (the "applied value of J") and the resultant driving force on the shock. Thus in general, $J_{tip} \neq J_\infty$. (This was also noted by Silling[15].) In certain exceptional cases, for example if the deformation is such that f = constant on C, the integral in (6.8) will vanish and then $J_{tip} = J_\infty$. The value of the shock driving traction f depends on (and is determined by) the particular kinetic relation governing the evolution of the shock. If the resultant driving force on the shock is in the positive x_1-direction then, by (6.8), $J_{tip} < J_\infty$.

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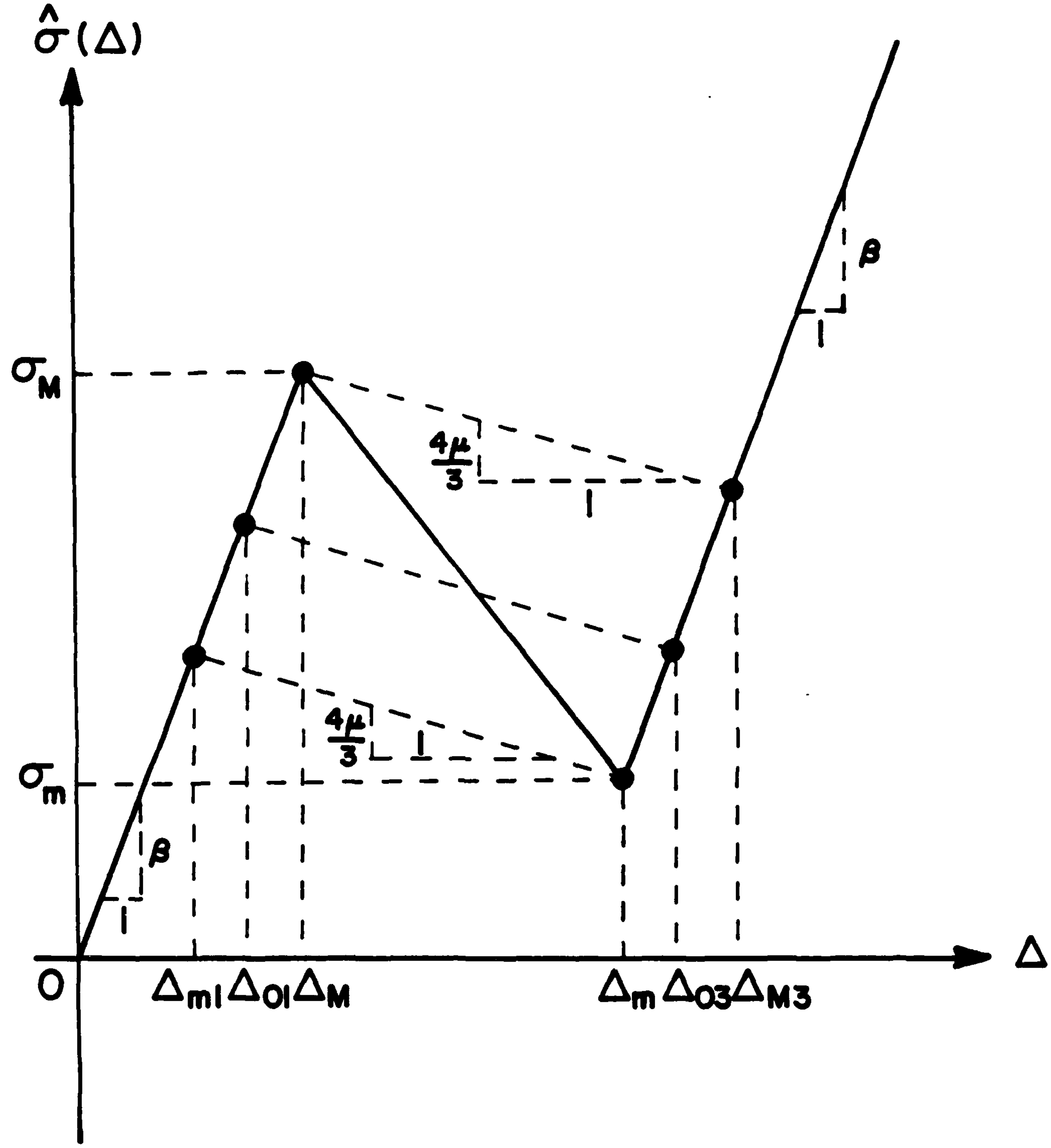

Figure 1. Stress response curve in pure dilatation.

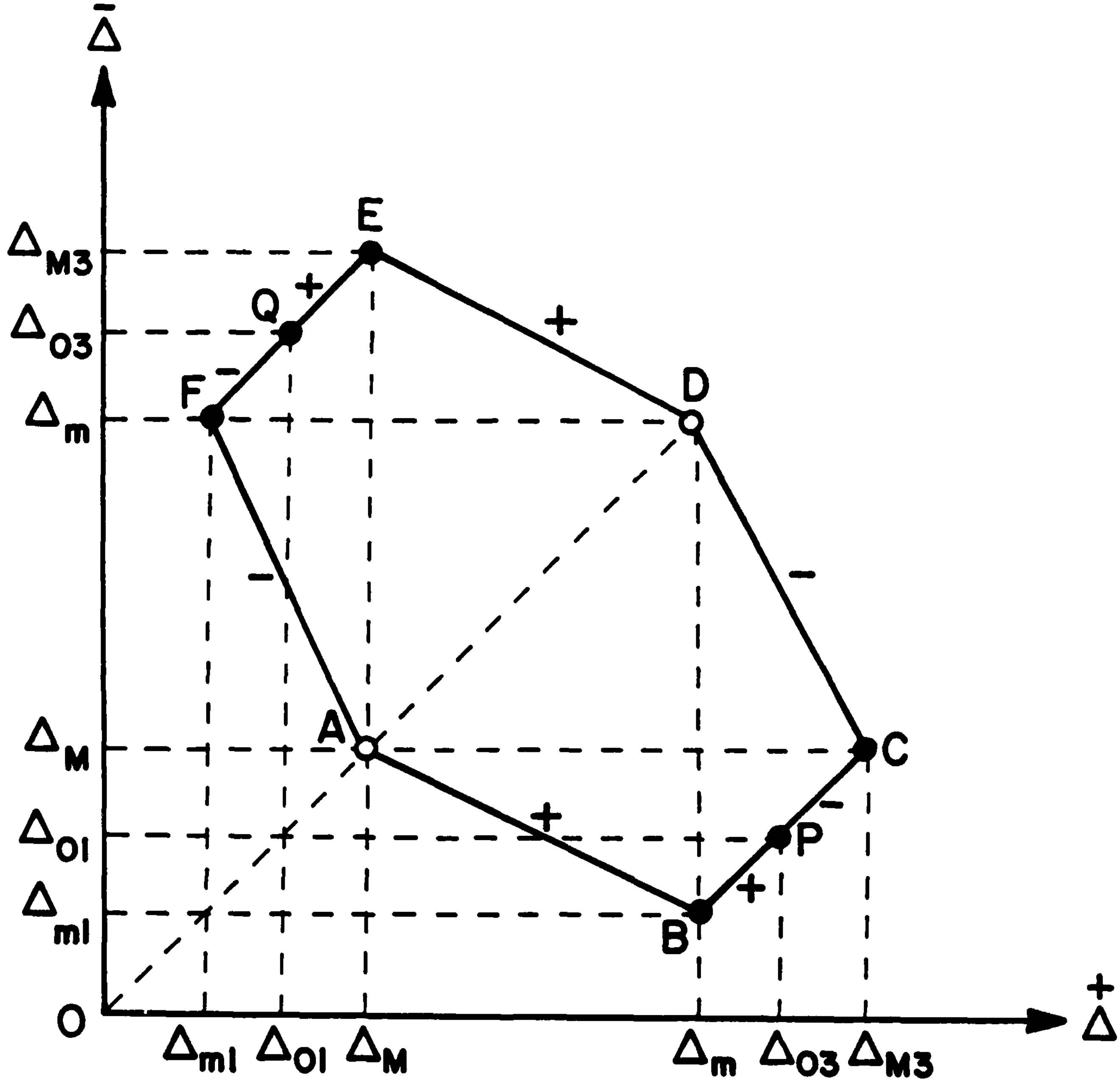

Figure 2. The set Ξ characterizing all possible shock states.

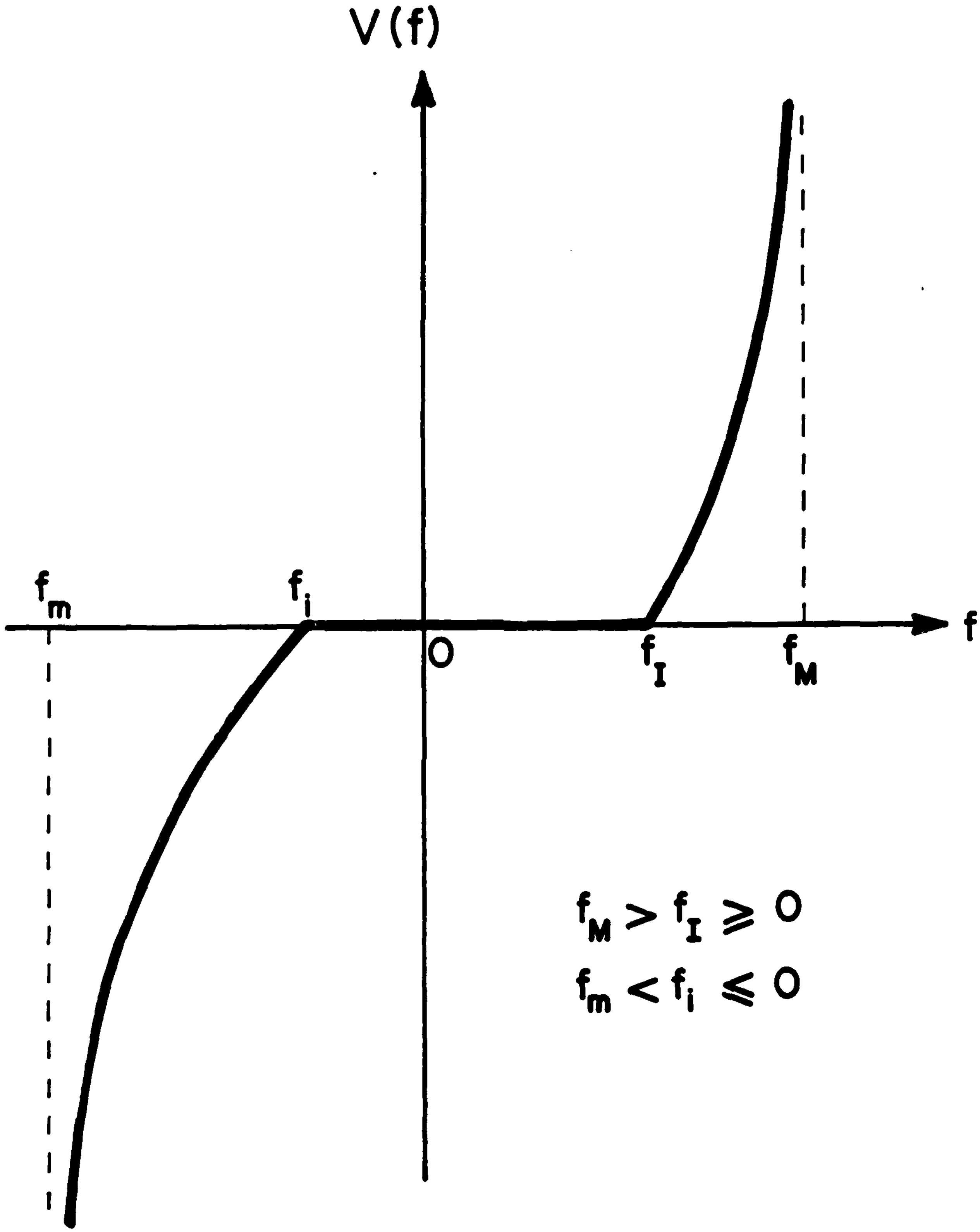

Figure 3. Kinetic response function.

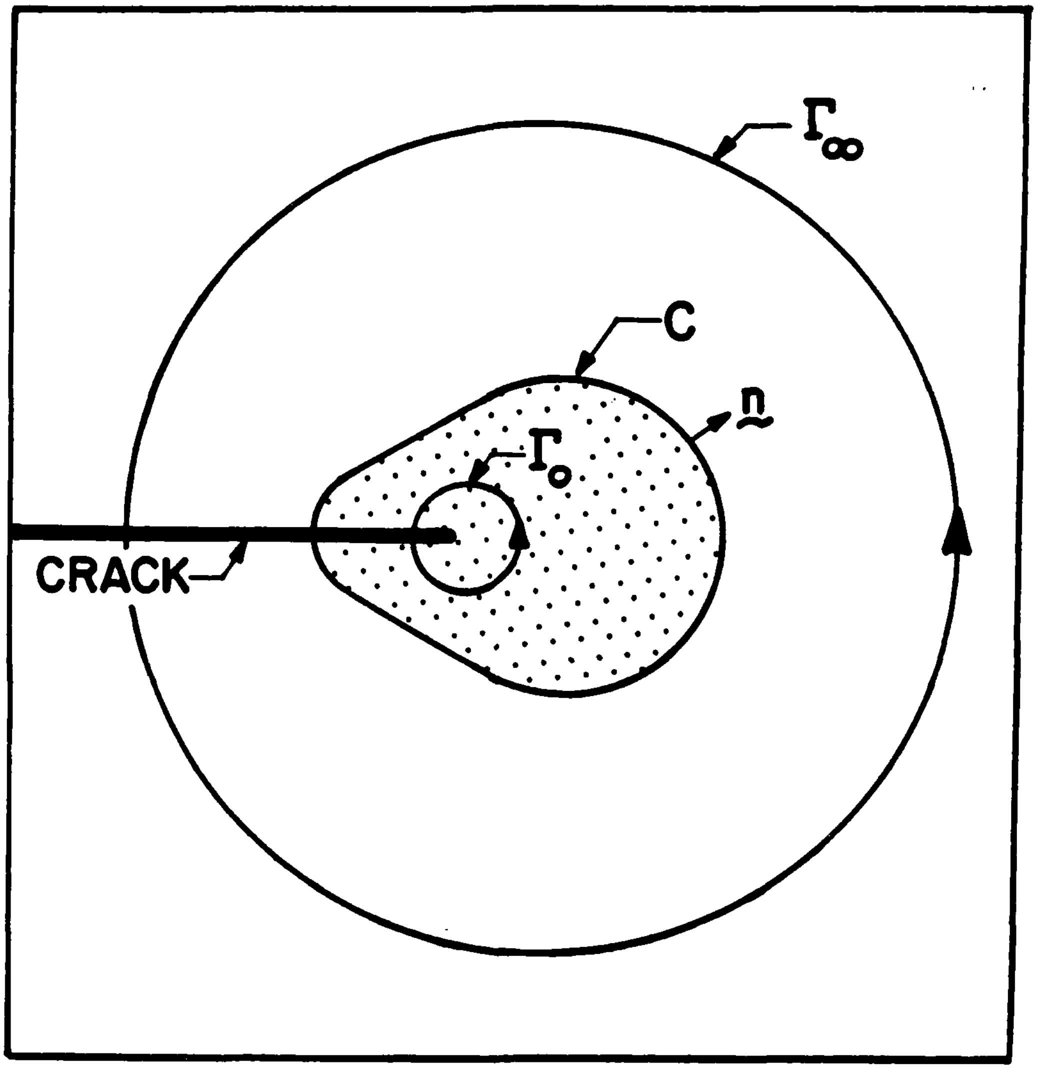

Figure 4. Geometry of cracked slab with phase boundary C and integration paths Γ_0, Γ_∞ .

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This paper, which is the first in a two-part study, addresses certain issues concerning the small-strain theory of nonlinear elasticity. It considers isotropic materials which possess a linear response in shear and a nonlinear response in dilatation, and (i) establishes an explicit necessary and sufficient condition for the existence of piecewise homogeneous deformations, (ii) obtains a characterization of the set of all such deformations, (iii) derives an expression for the "driving traction" on a surface of discontinuity in the strain, and finally, (iv) discusses the notion of a kinetic law. While the analysis is carried out within a three-dimensional setting, the results are shown to have a particularly simple form when expressed in terms of a certain constitutive function $\Sigma(\varepsilon)$. In part II of this study we examine a specific boundary-value problem.

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